Derivation of the model's log-linearization process in Chapter 5 (RBC) of Romer's textbook

Campbell (1994)

$$W_t = (1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t \quad ①$$

$$r_t = \alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} - \delta \quad ②$$

$$\frac{1}{c_t} = e^{-\rho} E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right] \quad ③$$

$$\frac{c_t}{1-l_t} = \frac{W_t}{b} \quad ④$$

$$K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t \quad ⑤$$

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t} \quad , \quad -1<\rho_A<1 \quad ⑥$$

$$\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{G,t} \quad , \quad -1<\rho_G<1 \quad ⑦$$

Intuition: Find the steady-state relationships then linearize equations by approximating about the logarithmic terms of the steady-state variables. As a result, the decision rules $C_t(K_t, A_t, G_t)$ and $L_t(K_t, A_t, G_t)$, or say $\tilde{C}_t(\tilde{K}_t, \tilde{A}_t, \tilde{G}_t)$ and $\tilde{L}_t(\tilde{K}_t, \tilde{A}_t, \tilde{G}_t)$, are linear. So is the capital evolving equation $\tilde{K}_{t+1}(\tilde{K}_t, \tilde{A}_t, \tilde{G}_t)$.

Denote by Z_t^* the steady-state value of variable Z in period t. So $\ln Z_t = \ln Z_t^* + \tilde{Z}_t$.

$$C_t = N_t \cdot c_t \,,\; L_t = N_t \cdot l_t \implies \tilde{C}_t = \tilde{c}_t \,,\; \tilde{L}_t = \tilde{l}_t$$

STEP 1 Find steady-state relationships

Consider a certainty version of the model, i.e., $\tilde{A}_t = \tilde{G}_t \equiv 0$.
Obviously, equations ① – ⑤ should hold along the balanced growth path.

Consider ③, now uncertainty disappears:

$$\frac{C^*_{t+1}}{C^*_t} = e^{-\rho}(1+r^*_{t+1}) \Longrightarrow 1+g = e^{-\rho}(1+r^*)$$

Taking logarithms on both sides, $\log(1+g) \simeq g$, $\log(1+r^*) \simeq r^*$

so $r^* = g + \rho$

$$② \Longrightarrow \frac{A^*_t L^*_t}{K^*_t} = \left(\frac{r^*_t + \delta}{\alpha}\right)^{\frac{1}{1-\alpha}} = \left(\frac{g+\rho+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \qquad ⑧$$

Insterting ⑧ into the production function yields:

$$\frac{Y^*_t}{K^*_t} = \left(\frac{A^*_t L^*_t}{K^*_t}\right)^{1-\alpha} = \frac{g+\rho+\delta}{\alpha} \qquad ⑨$$

Now rewrite ⑤:

$$\frac{K^*_{t+1}}{K^*_t} - (1-\delta) = \frac{Y^*_t}{K^*_t} - \frac{C^*_t}{K^*_t} - \frac{G^*_t}{K^*_t}$$

$$\Downarrow$$

$$n + g + \delta = \frac{Y^*_t}{K^*_t}\left(1 - \frac{C^*_t}{Y^*_t} - \frac{G^*_t}{Y^*_t}\right)$$

Define $\phi \equiv \exp\left[\bar{G} - (\bar{K})^{\alpha}(\bar{A}\bar{N}\ell^{*})^{1-\alpha}\right] = \dfrac{\exp(\bar{G} + (n+g)t)}{\exp(\bar{Y} + (n+g)t)} = \dfrac{G_t^{*}}{Y_t^{*}}$ (10)

where ℓ^{*} is the constant steady-state working effort, whose value we will figure out later.

Now (5) becomes

$$n+g+\delta = \frac{g+\rho+\delta}{\alpha}\left(1 - \frac{C_t^{*}}{Y_t^{*}} - \phi\right)$$

$$\Rightarrow \frac{C_t^{*}}{Y_t^{*}} = 1 - \phi - \frac{\alpha(n+g+\delta)}{g+\rho+\delta} \qquad (11)$$

Combining (1) and (4) yields:

$$1-\ell^{*} = b\cdot\frac{C_t^{*}}{w_t^{*}} = b\cdot\frac{N_t C_t^{*}}{N_t w_t^{*}} = b\cdot\frac{C_t^{*}}{\frac{1}{\ell^{*}}\cdot L_t w_t} = \frac{b}{1-\alpha}\frac{C_t^{*}}{Y_t^{*}}\cdot\ell^{*}$$

$$\Rightarrow \ell^{*} = 1\Big/\left[1 + \frac{b(1-\phi)}{1-\alpha} - \frac{b\alpha(n+g+\delta)}{(1-\alpha)(g+\rho+\delta)}\right] \qquad (12)$$

STEP 2 Take log-linearization and approximation

We know that:

(1) Equations ①-⑤ should still hold after adding shocks into the model.

(2) The observed time series of macro variables are fluctuating (closely) about their steady-state trends.

As a result:

(1) We would like to approximate these equations by taking first-order Taylor expansions on both sides. By doing so, we separate the fluctuation terms from the trends, as well as eliminate the non-linearity of the system.

(2) We rephrase macro variables in forms of their logarithms, as we care about the percentages of fluctuations rather then their magnitudes, in both theory and empirics. So we often need to take logarithms of both sides before approximating them.

Combine labor-market-clearing conditions ① and ④:

$$\frac{C_t}{1-l_t} = \frac{(1-\alpha) K_t^{\alpha} \cdot L_t^{-\alpha} \cdot A_t^{1-\alpha}}{b}$$

take logarithms: $\ln C_t - \ln(1-l_t) = \ln\left(\frac{1-\alpha}{b}\right) + \alpha \ln K_t - \alpha \ln L_t + (1-\alpha)\ln A_t$

$$\ln C_t - \ln(1-\exp(\ln l_t)) = \ln\left(\frac{1-\alpha}{b}\right) + \alpha \ln K_t - \alpha \ln L_t + (1-\alpha)\ln A_t$$

first-order Taylor expansion:

$$LHS(\ln C_t, \ln l_t) \approx LHS(\ln C_t^*, \ln l^*) + \frac{\partial LHS}{\partial \ln C_t}\Big|_{\substack{\ln C_t^* \\ \ln l^*}} \cdot \tilde{c}_t + \frac{\partial LHS}{\partial \ln l_t}\Big|_{\substack{\ln C_t^* \\ \ln l^*}} \cdot \tilde{l}_t$$

$$RHS(\ln K_t, \ln L_t, \ln A_t) \approx RHS(\ln K_t^*, \ln L_t^*, \ln A_t^*) + \frac{\partial RHS}{\partial \ln K_t}\Big|_{\substack{\ln K_t^* \\ \ln L_t^* \\ \ln A_t^*}} \cdot \tilde{K}_t + \frac{\partial RHS}{\partial \ln L_t}\Big|_{\substack{\ln K_t^* \\ \ln L_t^* \\ \ln A_t^*}} \cdot \tilde{L}_t$$

$$+ \frac{\partial RHS}{\partial \ln A_t}\Big|_{\substack{\ln K_t^* \\ \ln L_t^* \\ \ln A_t^*}} \cdot \tilde{A}_t$$

We know that $LHS(\ln C_t, \ln l_t) = RHS(\ln K_t, \ln L_t, \ln A_t)$

we further know $LHS(\ln C_t^*, \ln l^*) = RHS(\ln K_t^*, \ln L_t^*, \ln A_t^*)$.

i.e., general equilibrium must hold for the economy, with or without shocks.

So $LHS(\ln C_t, \ln l_t) - LHS(\ln C_t^*, \ln l^*) = RHS(\ln K_t, \ln L_t, \ln A_t) - RHS(\ln K_t^*, \ln L_t^*, \ln A_t^*)$

$$\Downarrow$$

$$\tilde{c}_t + \frac{l^*}{1-l^*} \cdot \tilde{l}_t = \alpha \cdot \tilde{K}_t - \alpha \tilde{L}_t + (1-\alpha)\tilde{A}_t$$

Recall $\tilde{c}_t = \tilde{C}_t$, $\tilde{l}_t = \tilde{L}_t \Rightarrow$

$$\tilde{C}_t = \alpha \tilde{K}_t + (1-\alpha)\tilde{A}_t - \left(\alpha + \frac{l^*}{1-l^*}\right)\tilde{L}_t$$

$$\tilde{C}_t = \lambda_1 \tilde{K}_t + \lambda_2 \tilde{A}_t + \lambda_3 \tilde{L}_t \qquad (13)$$

where $\lambda_1 = \alpha$, $\lambda_2 = (1-\alpha)$, $\lambda_3 = -\left(\alpha + \frac{l^*}{1-l^*}\right)$

Now consider (5):

$$\frac{K_{t+1}}{K_t} - (1-\delta) = \frac{Y_t}{K_t}\left(1 - \frac{C_t}{Y_t} - \frac{G_t}{Y_t}\right)$$

take logarithms $\Longrightarrow$

$$\ln[\exp(\ln k_{t+1}-\ln k_t)-(r\delta)] = \ln Y_t - \ln K_t + \ln[1-\exp(\ln C_t - \ln Y_t) - \exp(\ln G_t - \ln Y_t)]$$

$$LHS(\ln K_{t+1} - \ln K_t) = RHS(\ln Y_t - \ln K_t,\ \ln C_t - \ln Y_t,\ \ln G_t - \ln Y_t)$$

first-order Taylor expansion:

$$LHS(\ln K_{t+1} - \ln K_t) - LHS(\ln K^*_{t+1} - \ln K^*_t)$$

$$\approx \frac{d\,LHS}{d(\ln K_{t+1} - \ln K_t)}\Bigg|_{(\ln K^*_{t+1} - \ln K^*_t)} \cdot (\tilde{K}_{t+1} - \tilde{K}_t)$$

$\frac{K^*_{t+1}}{K^*_t} = 1+n+g$

$$= \frac{1+n+g}{n+g+\delta}(\tilde{K}_{t+1} - \tilde{K}_t)$$

Recall $\ln Y^*_t - \ln K^*_t = \ln(g+\rho+\delta) - \ln(\alpha)$

$$\ln C^*_t - \ln Y^*_t = \ln\left[1-\phi - \frac{\alpha(n+g+\delta)}{g+\rho+\delta}\right]$$

$$\ln G^*_t - \ln Y^*_t = \ln\phi$$

$$\Rightarrow RHS\left(\ln\frac{Y_t}{K_t}, \ln\frac{C_t}{Y_t}, \ln\frac{G_t}{Y_t}\right) - RHS\left(\ln\frac{Y^*_t}{K^*_t}, \ln\frac{C^*_t}{Y^*_t}, \ln\frac{G^*_t}{Y^*_t}\right)$$

$$= \tilde{Y}_t - \tilde{K}_t + \frac{\frac{\alpha(n+g+\delta)}{g+\rho+\delta}+\phi-1}{\frac{\alpha(n+g+\delta)}{g+\rho+\delta}}(\tilde{C}_t - \tilde{Y}_t) + \frac{-\phi}{\frac{\alpha(n+g+\delta)}{g+\rho+\delta}}(\tilde{G}_t - \tilde{Y}_t)$$

$$= \frac{g+\rho+\delta}{\alpha(n+g+\delta)}\cdot\tilde{Y}_t - \tilde{K}_t + \frac{\alpha(n+g+\delta)+(\phi-1)(g+\rho+\delta)}{\alpha(n+g+\delta)}\tilde{C}_t$$

$$- \frac{\phi(g+\rho+\delta)}{\alpha(n+g+\delta)}\tilde{G}_t$$

Equating the deviations of both sides $\Rightarrow$

$$\tilde{K}_{t+1} = \frac{g+\rho+\delta}{\alpha(1+n+g)}\tilde{Y}_t + \frac{1-\delta}{1+n+g}\tilde{K}_t + \frac{\alpha(n+g+\delta)+(\phi-1)(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{C}_t$$

$$- \frac{\phi(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{G}_t$$

$\tilde{Y}_t = \alpha\tilde{K}_t + (1-\alpha)(\tilde{L}_t + \tilde{A}_t)$

$$= \frac{1+g+\rho}{1+n+g}\tilde{K}_t + \frac{(1-\alpha)(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{A}_t + \frac{\alpha(n+g+\delta)+(\phi-1)(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{C}_t$$

$$- \frac{\phi(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{G}_t + \frac{(1-\alpha)(g+\rho+\delta)}{\alpha(1+n+g)}\tilde{L}_t$$

$$\tilde{K}_{t+1} = \lambda_4\tilde{K}_t + \lambda_5\tilde{A}_t + \lambda_6\tilde{C}_t + \lambda_7\tilde{G}_t + \lambda_8\tilde{L}_t \quad (14)$$

where $\lambda_4 = \frac{1+g+\rho}{1+n+g}$, $\lambda_5 = \frac{(1-\alpha)(g+\rho+\delta)}{\alpha(1+n+g)}$, $\lambda_6 = \frac{\alpha(n+g+\delta)+(\phi-1)(g+\rho+\delta)}{\alpha(1+n+g)}$

$\lambda_7 = -\frac{\phi(g+\rho+\delta)}{\alpha(1+n+g)}$, $\lambda_8 = \frac{(1-\alpha)(g+\rho+\delta)}{\alpha(1+n+g)}$

Take a detour :

- $\lambda_5 = \lambda_8$ in this model, but not necessarily in other cases.
- partial verification : $\lambda_4 + \lambda_6 + \lambda_7 + \lambda_8 = 1$.
  intuition : production function is CRS !

$$\lambda_4+\lambda_6+\lambda_7+\lambda_8=\frac{\alpha(1+g+\rho)+\alpha(n+g+\delta)-(1-\phi)(g+\rho+\delta)-\phi(g+\rho+\delta)+(1-\alpha)(g+\rho+\delta)}{\alpha(1+n+g)}$$

$$=\frac{\alpha(1+g+\rho+n+g+\delta-g-\rho-\delta)}{\alpha(1+n+g)}$$

$$=1$$

We treat ② and ③ sequentially.

$$② \Rightarrow \quad 1+r_t=\alpha\left(\frac{A_tL_t}{K_t}\right)^{1-\alpha}+1-\delta$$

take logarithms of both sides $\Rightarrow$

$$r_t=\ln\left[\exp\left(\ln\alpha+(1-\alpha)\cdot\ln\frac{A_tL_t}{K_t}\right)+1-\delta\right]$$

$$\Downarrow \quad \frac{A_t^*L_t^*}{K_t^*}=\left(\frac{g+\rho+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}}$$

$$r_t-r^*=\frac{(g+\rho+\delta)(1-\alpha)}{1+g+\rho}\left(\hat{A}_t+\hat{L}_t-\hat{k}_t\right) \qquad \widetilde{②}$$

Now consider ③ :

$$\frac{1}{c_t}=e^{-\rho}E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right]$$

$$\Downarrow$$

$$\frac{e^{\rho}}{(1+n)\breve{C}_t} = E_t\left[\frac{1}{\breve{C}_{t+1}}(1+r_{t+1})\right]$$

$$\Downarrow$$

$$\frac{e^{\rho}}{1+n}\cdot \exp(-\ln \breve{C}_t) = E_t\left[\exp(-\ln \breve{C}_{t+1})(1+r_{t+1})\right]$$

$$LHS(\ln \breve{C}_t) - LHS(\ln \breve{C}_t^*)$$

$$\approx -\frac{e^{\rho}}{(1+n)\breve{C}_t^*}\tilde{C}_t$$

Define $RHS' = \exp(-\ln \breve{C}_{t+1})(1+r_{t+1}) \approx \exp(-\ln \breve{C}_{t+1} + r_{t+1})$

$$RHS'(\ln \breve{C}_{t+1}, r_{t+1}) - RHS'(\ln \breve{C}_{t+1}^*, r_{t+1}^*)$$

$$\approx -\frac{1+g+\rho}{\breve{C}_{t+1}^*}\cdot\tilde{C}_{t+1} + \frac{1+g+\rho}{\breve{C}_{t+1}^*}(r_{t+1} - r^*)$$

$$LHS(\ln \breve{C}_t) = E_t\left[RHS'(\ln \breve{C}_{t+1}, r_{t+1})\right]$$

$$LHS(\ln \breve{C}_t^*) = E_t(RHS'(\ln \breve{C}_{t+1}^*, r_{t+1}^*))$$

$$\Rightarrow -\frac{e^{\rho}}{(1+n)\breve{C}_t^*}\tilde{C}_t = E_t\left[-\frac{1+g+\rho}{\breve{C}_{t+1}^*}\tilde{C}_{t+1} + \frac{1+g+\rho}{\breve{C}_{t+1}^*}(r_{t+1} - r^*)\right]$$

$$\Downarrow$$

$$E_t\tilde{C}_{t+1} - \frac{e^{\rho}(1+n+g)}{(1+g+\rho)(1+n)}\tilde{C}_t = E_t(r_{t+1} - r^*)$$

$$\Downarrow \quad e^{n+g} \approx 1+n+g,\ e^{g+\rho} \approx 1+g+\rho,\ e^{n} \approx 1+n$$

$$E_t(\tilde{C}_{t+1} - \tilde{C}_t) = E_t(r_{t+1} - r^*)$$

Substituting $\tilde{②}$ into the equation above $\Rightarrow$

$$E_t(\tilde{C}_{t+1} - \tilde{C}_t) = \lambda_q E_t(\tilde{A}_{t+1} + \tilde{L}_{t+1} - \tilde{k}_{t+1}) \quad (15)$$

where $\lambda_q = \dfrac{(g+\rho+\delta)(1-\alpha)}{1+g+\rho}$

Take a detour, the Euler equation can also be treated in this way:

Define $R_t = e^{r_t}$.

Rewrite the Euler equation as:

$$\frac{e^{\rho}}{(1+n)C_t} = E_t\left[\frac{1}{C_{t+1}} R_{t+1}\right]$$

In this way, $\frac{1}{C_{t+1}}$ and R_{t+1} are jointly log-normal at time t

For linear system (linear optimal rules and evolving equations), we have homoscedasticity property:

$$\left.\begin{pmatrix} \ln(\frac{1}{C_{t+1}}) \\ \ln(R_{t+1}) \end{pmatrix}\right| t \sim N\left[\begin{matrix} E_t[\ln(\frac{1}{C_{t+1}})] \\ E_t[\ln(R_{t+1})] \end{matrix}, \begin{pmatrix} Var_c & Cov_{c,r} \\ Cov_{c,r} & Var_r \end{pmatrix}\right]$$

The covarian matrix is time-invariant

(randomness in them are from $\varepsilon_{G,t+1}$ and $\varepsilon_{A,t+1}$, which enter in linear ways)

Take logarithms of both sides $\Longrightarrow$

If X is log-normal then $\log(EX) = E(\ln X) + \frac{1}{2}Var(\ln X)$

$$\ln(e^{\rho}/(1+n)) - \ln C_t = E_t[\ln(R_{t+1}/C_{t+1})] + \frac{1}{2}Var_t[\ln(R_{t+1}/C_{t+1})]$$

$$\Rightarrow -\ln C_t' = E_t(\ln R_{t+1} - \ln G_{t+1}) - \ln(e'/(1+n)) + \frac{1}{2}(Var_c + Var_r + 2\,Cov_{c,r})$$

In a LINEAR system, if a real number $\ln C_t$ and two random variables $\ln R_{t+1}$, $\ln G_{t+1}$ satisfies the equation above, so do $\ln C_t' = \ln C_t + \tilde{C}_t$, $\ln R_{t+1}' = \ln R_{t+1} + \tilde{R}_{t+1}$, and $\ln G_{t+1}' = \ln G_{t+1} + \tilde{G}_{t+1}$

$$\Rightarrow E_t(\tilde{C}_{t+1} - \tilde{C}_t) = E_t(r_{t+1} - r^*)$$

Now the equilibrium dynamics becomes:

$$\tilde{C}_t = \lambda_1 \tilde{K}_t + \lambda_2 \tilde{A}_t + \lambda_3 \tilde{L}_t \quad (13)$$

$$\tilde{K}_{t+1} = \lambda_4 \tilde{K}_t + \lambda_5 \tilde{A}_t + \lambda_6 \tilde{C}_t + \lambda_7 \tilde{G}_t + \lambda_8 \tilde{L}_t \quad (14)$$

$$E_t(\tilde{C}_{t+1} - \tilde{C}_t) = \lambda_9 E_t(\tilde{A}_{t+1} + \tilde{L}_{t+1} - \tilde{K}_{t+1}) \quad (15)$$

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t} \quad (6)$$

$$\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{G,t} \quad (7)$$

STEP 3 method of undetermined coefficients.

In the linear system, it is natural to assume that all decision rules are linear.

following the treatment in textbook:

$$\tilde{C}_t \simeq a_{CK} \tilde{K}_t + a_{CA} \tilde{A}_t + a_{CG} \tilde{G}_t \quad (16)$$

$$\tilde{L}_t \simeq a_{LK} \tilde{K}_t + a_{LA} \tilde{A}_t + a_{LG} \tilde{G}_t \quad (17)$$

$$\tilde{K}_{t+1} \approx b_{KK}\tilde{K}_t + b_{KA}\tilde{A}_t + b_{KG}\tilde{G}_t \quad (18)$$

Substituting (16), (17), (18) into (13) yields:

$$a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t = (\lambda_1 + \lambda_3 a_{LK})\tilde{K}_t + (\lambda_2 + \lambda_3 a_{LA})\tilde{A}_t + \lambda_3 a_{LG}\tilde{G}_t$$

the equality holds for each t $\Rightarrow$

$$\begin{aligned} a_{CK} &= \lambda_1 + \lambda_3 a_{LK} \\ a_{CA} &= \lambda_2 + \lambda_3 a_{LA} \\ a_{CG} &= \lambda_3 \cdot a_{LG} \end{aligned} \qquad \text{(eqset 1)}$$

Now for (14) $\Rightarrow$

$$\begin{aligned} b_{KK}\tilde{K}_t + b_{KA}\tilde{A}_t + b_{KG}\tilde{G}_t = &(\lambda_4 + \lambda_6 a_{CK} + \lambda_8 \cdot a_{LK})\tilde{K}_t \\ &+ (\lambda_5 + \lambda_6 a_{CA} + \lambda_8 a_{LA})\tilde{A}_t \\ &+ (\lambda_7 + \lambda_6 a_{CG} + \lambda_8 a_{LG})\tilde{G}_t \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} b_{KK} &= \lambda_4 + \lambda_6 a_{CK} + \lambda_8 \cdot a_{LK} \\ b_{KA} &= \lambda_5 + \lambda_6 a_{CA} + \lambda_8 a_{LA} \\ b_{KG} &= \lambda_7 + \lambda_6 a_{CG} + \lambda_8 a_{LG} \end{aligned} \qquad \text{(eqset 2)}$$

Now for ⑮

Note that: $E_t \widetilde{A}_{t+1} = \rho_A \widehat{A}_t$, $E_t \widetilde{G}_{t+1} = \rho_G \cdot \widetilde{G}_t$

$$LHS = E_t\left[a_{CK}(\widetilde{K}_{t+1} - \widetilde{K}_t) + a_{CA}(\widetilde{A}_{t+1} - \widehat{A}_t) + a_{CG}(\widetilde{G}_{t+1} - \widetilde{G}_t)\right]$$

$$= a_{CK}(b_{KK}-1)\widetilde{K}_t + a_{CK} b_{KA} \widehat{A}_t + a_{CK} b_{KG} G_t + a_{CA}(\rho_A - 1)\widetilde{A}_t + a_{CG}(\rho_G - 1)\widetilde{G}_t$$

$$= a_{CK}(b_{KK}-1)\widetilde{K}_t + [a_{CK} b_{KA} + a_{CA}(\rho_A - 1)]\widetilde{A}_t + [a_{CK} b_{KG} + a_{CG}(\rho_G - 1)]\widetilde{G}_t$$

$$RHS = \lambda q \rho_A \widetilde{A}_t + \lambda q E_t\left[(a_{LK}-1)\widetilde{K}_{t+1} + a_{LA}\widetilde{A}_{t+1} + a_{LG} \widetilde{G}_{t+1}\right]$$

$$= \lambda q \rho_A (1 + a_{LA})\widehat{A}_t + \lambda q\, a_{LG} \rho_G \cdot \widetilde{G}_t$$

$$+ \lambda q (a_{LK} - 1)(b_{KK}\widehat{K}_t + b_{KA}\widetilde{A}_t + b_{KG}\widetilde{G}_t)$$

$$= \lambda q (a_{LK}-1) b_{KK} \widetilde{K}_t + [\lambda q \rho_A (1 + a_{LA}) + \lambda q (a_{LK}-1) b_{KA}]\widetilde{A}_t$$

$$+ [\lambda q\, a_{LG} \rho_G + \lambda q (a_{LK}-1) b_{KG}]\widetilde{G}_t$$

$$\Rightarrow a_{CK}(b_{KK}-1) = \lambda q (a_{LK}-1) b_{KK}$$

$$a_{CK} b_{KA} + a_{CA}(\rho_A - 1) = \lambda q \rho_A (1 + a_{LA}) + \lambda q (a_{LK}-1) b_{KA}$$ (eq set 3)

$$a_{CK} b_{KG} + a_{CG}(\rho_G - 1) = \lambda q\, a_{LG} \rho_G + \lambda q (a_{LK}-1) b_{KG}$$

STEP 4 Calibration

value	matching method
$\alpha = \frac{1}{3}$	1 − labor share
$g = 0.5\%$	(long-run TFP growth rate)/4
$n = 0.25\%$	population data
$\delta = 2.5\%$	annual depreciation rate for capital is 10% (accounting rule)
$\rho_A = 0.95$	Estimate A_t, detrend to get $\tilde{A}_t$, use AR(1) to fit data
$\rho_G = 0.95$	observe G_t, detrend to get $\hat{G}_t$, use AR(1) to fit data.
$\phi = 0.2$	in the long run, $G^*/Y^* = 20\%$
$\rho = 1\%$	such that $r^* = \rho + g = 1.5\%$
$b = 2.519685$	such that $\ell^* = 1/3$, 8 hours spent at work per day.
Var_{ε_A} Var_{ε_G}	Estimate $\tilde{A}_t$ and $\hat{G}_t$, fit data using AR(1) needed in simulations. not in predictions because the system is linear.

$$
\left\{
\begin{aligned}
a_{CK} &= \lambda_1 + \lambda_3 a_{LK} && (19)\\
a_{CA} &= \lambda_2 + \lambda_3 a_{LA} && (20)\\
a_{CG} &= \lambda_3 \cdot a_{LG} && (21)\\
b_{KK} &= \lambda_4 + \lambda_6 a_{CK} + \lambda_8 \cdot a_{LK} && (22)\\
b_{KA} &= \lambda_5 + \lambda_6 a_{CA} + \lambda_8 a_{LA} && (23)\\
b_{KG} &= \lambda_7 + \lambda_6 a_{CG} + \lambda_8 a_{LG} && (24)\\
a_{CK}(b_{KK} - 1) &= \lambda_9 (a_{LK} - 1) b_{KK} && (25)\\
a_{CK} b_{KA} + a_{CA}(\rho_A - 1) &= \lambda_9 \rho_A (1 + a_{LA}) + \lambda_9 (a_{LK} - 1) b_{KA} && (26)\\
a_{CK} b_{KG} + a_{CG}(\rho_G - 1) &= \lambda_9 a_{LG} \rho_G + \lambda_9 (a_{LK} - 1) b_{KG} && (27)
\end{aligned}
\right.
$$

Combine (19), (22), , we get:

$$
\begin{aligned}
b_{KK} &= \lambda_4 + \lambda_1 \lambda_6 + (\lambda_3 \lambda_6 + \lambda_8) a_{LK}. \\
&= Q_0 + Q_1 a_{LK}.
\end{aligned}
$$

where $Q_0 = \lambda_4 + \lambda_1 \lambda_6$, $Q_1 = \lambda_3 \lambda_6 + \lambda_8$.

Substituting into (25) yields

$$[\lambda_1 + (\lambda_3 - \lambda_9) a_{LK} + \lambda_9](Q_0 + Q_1 a_{LK}) - \lambda_1 - \lambda_3 a_{LK} = 0$$

$$\Downarrow$$

$$\check{Q}_2 a_{Lk}^2 + Q_3 a_{Lk} + Q_4 = 0$$

$$a_{Lk} \in \left\{ \frac{-Q_3 + \sqrt{Q_3^2 - 4Q_2Q_4}}{2Q_2}, \frac{-Q_3 - \sqrt{Q_3^2 - 4Q_2Q_4}}{2Q_2} \right\}$$

where $Q_2 = Q_1(\lambda_3 - \lambda_9)$,

$$Q_3 = (\lambda_3 - \lambda_9) \cdot Q_0 + (\lambda_1 + \lambda_9) \cdot Q_1 - \lambda_3$$

$$Q_4 = (\lambda_1 + \lambda_9) Q_0 - \lambda_1$$

With all the calibrated values,

if $a_{Lk} = \frac{-Q_3 - \sqrt{Q_3^2 - 4Q_2Q_4}}{2Q_2} = 0.6 \Rightarrow a_{Ck} = -0.17$

$$\Rightarrow \underline{b_{kk} = 1.0771}$$

Cannot be the case, otherwise a time t shock $\tilde{k}_t$ will be amplified over time, the system becomes unstable!

So $a_{Lk} = \frac{-Q_3 + \sqrt{Q_3^2 - 4Q_2Q_4}}{2Q_2} = -0.31$

$$a_{Ck} = \lambda_1 + \lambda_3 a_{Lk} = 0.59$$

$$b_{kk} = Q_0 + Q_1 a_{Lk} = 0.95$$

Given the values of a_{LK}, a_{CK}, and b_{KK}, the remaining part of the equation system becomes purely linear. It is thus straightforward to have:

$$\begin{cases} a_{LK} = -0.31 \\ a_{CK} = \lambda_1 + \lambda_3 a_{LK} = 0.59 \\ b_{KK} = Q_0 + Q_1 a_{LK} = 0.95 \\ a_{LA} = \dfrac{-\{[a_{CK} - \lambda_9(a_{LK}-1)](\lambda_5 + \lambda_2\lambda_6) + (\rho_A - 1)\lambda_2 - \lambda_9\rho_A\}}{[a_{CK} - \lambda_9(a_{LK}-1)](\lambda_3\lambda_6 + \lambda_8) + (\rho_A - 1)\lambda_3 - \lambda_9\rho_A} = 0.35 \\ a_{CA} = \lambda_2 + \lambda_3 a_{LA} = 0.38 \\ b_{KA} = \lambda_5 + \lambda_2\lambda_6 + (\lambda_3\lambda_6 + \lambda_8) a_{LA} = 0.08 \\ a_{LG} = \dfrac{-[a_{CK} - \lambda_9(a_{LK}-1)]\cdot\lambda_7}{[a_{CK} - \lambda_9(a_{LK}-1)](\lambda_3\lambda_6 + \lambda_8) + \lambda_3(\rho_G - 1) - \lambda_9\rho_G} = 0.15 \\ a_{CG} = \lambda_3 \cdot a_{LG} = -0.13 \\ b_{KG} = \lambda_7 + (\lambda_3\lambda_6 + \lambda_8) a_{LG} = -0.004 \end{cases}$$

All coefficients determined.
Values in textbook verified.

STEP5

The system is used to: (1) fit data (2) simulate (3) predict.

(1) Fit data.

Plosser (1989) Figures 2-6 for an example.

Figure out $\hat{A}_t$, $\hat{G}_t$, use the system to fit data.

Check slides. China economy fitting.

Compare predicted and observed macro variables based on observed $\hat{A}_t$'s and $\hat{G}_t$'s.

(2) Simulate the system to characterize the comovement characteristics of macro variables or to calculate the probability of a specific event.

See table 5.4 in textbook for an example.

(3) Prediction.

Use the linear system to predict the pure effects of some shock on the whole economy, over time.

Impulse Response Functions

L stands for lag operator hereafter

Now, rewrite (18) $\Rightarrow (1-b_{kk}L)\tilde{k}_{t+1} = b_{kA}\tilde{A}_t + b_{kG}\tilde{G}_t$

recall $(1-\rho_A L)\tilde{A}_t = \varepsilon_{A,t}$, $(1-\rho_G L)\tilde{G}_t = \varepsilon_{G,t}$.

Substituting equations above into (13) yields:

$$\tilde{C}_t = a_{CK}(1-b_{kk}L)^{-1} b_{kA}\tilde{A}_{t-1} + a_{CK}(1-b_{kk}L)^{-1} b_{kG}\tilde{G}_{t-1} + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t$$

$$\Downarrow$$

$$\tilde{C}_t = \frac{a_{CK}\, b_{kA}\, L}{(1-b_{kk}L)(1-\rho_A L)}\varepsilon_{A,t} + \frac{a_{CA}}{1-\rho_A L}\varepsilon_{A,t} +$$

$$\frac{a_{CK}\, b_{kG}\, L}{(1-b_{kk}L)(1-\rho_G L)}\varepsilon_{G,t} + \frac{a_{CG}}{1-\rho_G L}\varepsilon_{G,t}$$

$$\Downarrow$$

$$(1-b_{kk}L)(1-\rho_A L)(1-\rho_G L)\tilde{C}_t$$
$$= a_{CK}b_{kA}(1-\rho_G L)L\varepsilon_{A,t} + (1-b_{kk}L)(1-\rho_G L)\varepsilon_{A,t} +$$
$$a_{CK}b_{kG}(1-\rho_A L)L\varepsilon_{G,t} + (1-b_{kk}L)(1-\rho_A L)\varepsilon_{G,t}$$

$$\Downarrow$$

$$\tilde{C}_t = u_1\tilde{C}_{t-3} + u_2\tilde{C}_{t-2} + u_3\tilde{C}_{t-1} + u_4\varepsilon_{A,t} + u_5\varepsilon_{A,t-1} + u_6\varepsilon_{A,t-2}$$
$$+ u_7\varepsilon_{G,t} + u_8\varepsilon_{G,t-1} + u_9\varepsilon_{G,t-2}$$

Analogously $\Rightarrow$

$$\tilde{I}_t = \tau_1\tilde{I}_{t-3} + \tau_2\tilde{I}_{t-2} + \tau_3\tilde{I}_{t-1} + \tau_4\varepsilon_{A,t} + \tau_5\varepsilon_{A,t-1} + \tau_6\varepsilon_{A,t-2} + \tau_7\varepsilon_{G,t} + \tau_8\varepsilon_{G,t-1} + \tau_9\varepsilon_{G,t-2}.$$

Can you determine the τ_i's and u_i's ?

We often treat shocks separately since they are assumed to be uncorrelated.

replicate Figures 5.2-5.4 in textbook.

$\hat{G}_t \equiv 0$, $\hat{A}_1 = \varepsilon_{A,1} = 1\%$. $\varepsilon_{A,t} \equiv 0$ for $t \geq 2$.

So we can figure out the 1 percentage technology shock's pure effect over time.

(18) $\Rightarrow$ $\hat{k}_{t+1} = \dfrac{b_{kA}}{(1-b_{kk}L)(1-\rho_A L)} \varepsilon_{A,t}$.

(13) $\Rightarrow$ $\tilde{C}_t = \dfrac{b_{kA}a_{ck}L + a_{cA}(1-b_{kk}L)}{(1-b_{kk}L)(1-\rho_A L)} \varepsilon_{A,t}$

or say:

$\hat{A}_t = \rho_A^{t-1} \cdot \varepsilon_{A,1}$,

$\tilde{C}_t = (b_{kk}+\rho_A)\tilde{C}_{t-1} - b_{kk}\rho_A \cdot \tilde{C}_{t-2} + a_{cA}\varepsilon_{A,t} + (b_{kA}a_{ck} - a_{cA}b_{kk})\varepsilon_{A,t-1}$

$\tilde{L}_t = (b_{kk}+\rho_A)\tilde{L}_{t-1} - b_{kk}\cdot\rho_A \tilde{L}_{t-2} + a_{LA}\varepsilon_{A,t} + (b_{kA}a_{Lk} - a_{LA}b_{kk})\varepsilon_{A,t-1}$

$\tilde{k}_t = (b_{kk}+\rho_A)\tilde{k}_{t-1} - b_{kk}\rho_A \tilde{k}_{t-2} + b_{kA}\cdot\varepsilon_{A,t-1}$

$\tilde{Y}_t = [\alpha+(1-\alpha)a_{Lk}]\tilde{k}_t + (1-\alpha)(1+a_{LA})\tilde{A}_t$

$r_t - r^* = \lambda q(\tilde{A}_t + \tilde{L}_t - \tilde{k}_t)$ $\quad \tilde{w}_t = \tilde{Y}_t - \tilde{L}_t$.

ARMA(2, 1) processes.

Now replicate Figures 5.2, 5.3, 5.4

Do practice with codes!