Supplementary Notes on Chapter 2 of D. Romer's Advanced Macroeconomics Textbook (4th Edition)

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Calculus of variations (变分法)

- A field of mathematical analysis that deals with maximizing or minimizing **functionals**, which are mappings from a set of functions to the real numbers.
- Functionals are often expressed as definite integrals involving functions and their derivatives. (e.g., the famous shortest (in time) path problem)
- The **Euler-Lagrange equation** provides a **necessary condition** for finding extrema.

Euler-Lagrange equation

Intuition: Finding the extrema of functionals is similar to finding the maxima and minima of functions. This tool provides a link between them to solve the problem. Consider the functional

$$J[y] = \int_{x_1}^{x_2} L\left(x, y(x), y'(x)\right) \mathrm{d}x, \tag{1}$$

where

- x_1, x_2 are constants.
- $y(x)$ is twice continuously differentiable.
- $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$.
- $L\left(x, y(x), y'(x)\right)$ is twice continuously differentiable with respect to all arguments x, y, and y'.

Euler-Lagrange equation (Continued)

If $J[y]$ attains a local minimum at f, and $\eta(x)$ is an arbitrary function that has at least one derivative and vanishes at the endpoints x_1 and x_2, then for any number $\varepsilon \to 0$, we must have

$$J[f] \leq J[f + \varepsilon\eta]\,. \tag{2}$$

Term $\varepsilon\eta$ is called the **variation** of the function f. Now define

$$\Phi(\varepsilon) = J[f + \varepsilon\eta]. \tag{3}$$

Since $J[y]$ has a local minimum at $y = f$, it must be the case that $\Phi(\varepsilon)$ has a minimum at $\varepsilon = 0$ and thus

$$\Phi'(0) = \left.\frac{\mathrm{d}\Phi}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left.\frac{\mathrm{d}L}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{d}x = 0. \tag{4}$$

Euler-Lagrange equation (Continued)

Now taking total derivative of $L\left[x, f+\varepsilon\eta, (f+\varepsilon\eta)'\right]$, we have:

$$\frac{\mathrm{d}L}{\mathrm{d}\varepsilon} = \frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial y'}\eta'. \tag{5}$$

Inserting (5) into (4) gives us

$$\begin{aligned} 0 = \int_{x_1}^{x_2} \frac{\mathrm{d}L}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} &= \int_{x_1}^{x_2} \left(\frac{\partial L}{\partial f}\eta + \frac{\partial L}{\partial f'}\eta'\right)\mathrm{d}x \\ &= \int_{x_1}^{x_2} \left(\frac{\partial L}{\partial f}\eta - \eta\frac{\mathrm{d}(\frac{\partial L}{\partial f'})}{\mathrm{d}x}\right)\mathrm{d}x + \frac{\partial L}{\partial f'}\eta\bigg|_{x_1}^{x_2} \\ &= \int_{x_1}^{x_2} \eta\left(\frac{\partial L}{\partial f} - \frac{\mathrm{d}(\frac{\partial L}{\partial f'})}{\mathrm{d}x}\right)\mathrm{d}x, \end{aligned}$$

where the last lines uses integration by parts and the fact that η vanishes at x_1 and x_2.

Euler-Lagrange equation (Continued)

Now given

$$\int_{x_1}^{x_2} \eta \left(\frac{\partial L}{\partial f} - \frac{\mathrm{d}(\frac{\partial L}{\partial f'})}{\mathrm{d}x} \right) \mathrm{d}x = 0, \tag{6}$$

the **fundamental lemma of calculus of variations** makes sure that

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}(\frac{\partial L}{\partial f'})}{\mathrm{d}x} = 0 \tag{7}$$

must hold!

Euler-Lagrange equation (Continued)

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- However, it is possible to attain (7) based on (6) without applying the lemma!

Euler-Lagrange equation (Continued)

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- However, it is possible to attain (7) based on (6) without applying the lemma!
- A special form of η?
- How about $\eta(x)$ equals $-(x - x_1)(x - x_2) \left[\frac{\partial L}{\partial f} - \frac{\mathrm{d}(\frac{\partial L}{\partial f'})}{\mathrm{d}x} \right]$ for $x \in [x_1, x_2]$ and 0 for $x \notin [x_1, x_2]$?

Euler-Lagrange equation (Continued)

- How does (7) degenerate if y' is not an argument of L?
- Homework: based on equation (2.16) in your textbook, try to derive (2.17).

To be continued...